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Full Length Research Paper

Integrated versus non-integrated inventory management

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The increase in the variety of the products and size of the manufacturing organizations, has led to the changes in the organizational structure of companies. Dividing an organization into independent business units, based on the family products, is a common organizational design among companies. As one of the biggest Iranian dairy companies, KALEH Dairy Group has already implemented this design to increase its flexibility in a competitive environment for new product development. KALEH's problem now is making decision about products inventory management approach, because most of its products which are included in various groups have two roles. They have their own market demand. Meanwhile, they could act as raw material for some other products. In such condition, inventory management of final product can be done by two approaches. First one is independent inventory management by each business unit and the second one is integrated inventory management. This paper is aimed at showing how to help managers to choose appropriate mechanism for inventory management. To do so, a mathematical model is developed for integrated status and the results are compared with EPQ (Economic production quantity) model. The final results of mathematical model show that for products with two roles, the integrated inventory management imposes less costs on the company.

Key words: Inventory management, Business units, Economic production level.

INTRODUCTION

Organizational structure design based on family products is one of the common approaches in many companies (Daft, 2009). This kind of structure is now established and implemented in dairy companies widely. For instance "KALEH" dairy group formed some business units based on products category. In each unit new product development, production and final product sale are done as main functions. This business unit structure of the company has many benefits. In new product development area each unit expands the economic of scope based on market need in a way that products' variety is increased comparing to the old structure (Alam et al., 2010).

In production area, the waste is decreased and economy of scale happened because of improvement in management skills and specialized activities. But because of dependency between some products in business units, non integrated inventory management can cause increase in inventory costs (Alam, 2009). For example the product of yogurt business unit, is the raw material of other units. So to manage the inventory, two approaches can be utilized. One can be family inventory management by each business unit and the other is integrated dependent products management. In this paper these two approaches will be compared by developing mathematical model.

Literature review

In inventory management literature, the inventory systems have been classified based on the dependency and independency of the products demand. As some examples, Material Requirements Planning (MRP) is an inventory system when we have a dependent demand and Order Point System is the one for independent

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demand (Tersine, 1994). Each of aforementioned systems has been studied and developed by various researchers, some of which we will mentioned in this paper.

Material Requirements Planning (MRP) is an Information System which is used for Managing the inventory and scheduling the ordering of products with dependent demand (Jacobs and Chase, 2008). One of the main woes in MRP and other inventory systems is determining the order amount which is called “lot sizing determination”. The response to this question will be the input of MRP and production scheduling systems, when the demand is dependent and independent, respectively. A lot of work has been done in order to find an appropriate response to this question. First model of this kind is called Economical Order Quantity (EOQ) and referred to Harris. This model has been developed based on a fixed demand assumption (Harris, 1915). A simple expansion to EOQ, the Economic Production Quantity (EPQ) is reached. In this model, the product is assumed to be received or produced gradually and not at once (Nahmias, 2004). The aforementioned models have also been developed for the conditions when there is a backordering and shortage (W. J. Hopp and M. L. Spearman, 2000). In recent years, the researchers have made the EOQ model more appropriate for the real world by releasing some of its unrealistic assumptions.

Among them we can mention: Salameh and Jaber that omitted the assumption of equality of the quality of all received orders(Salameh and Jaber, 2000), Tsou that took in to account the quality costs (Tsou, 2005), Wee et al that considered the inequality of orders quality and also the shortage of orders simultaneously (Wee et al., 2005), He et al developed a production-inventory model for deteriorating items with multiple-market demand, where each market has a different selling season and a different constant demand rate (He et al., 2010), and Laguna et al presented a method to obtain the solution of the classic EOQ and EPQ models when the lot size must be an integer quantity (Laguna et al, 2010).

In many systems, the demand doesn’t present a monotonous behavior and varies from period to period. Due to their importance in industry and mathematical complexity, dynamic lot-sizing problems are frequently studied. Wagner and Whithin introduced a dynamic programming model in which minor demand is a function of time (Wagner and Whitin, 1958). Hwang et al developed Wagner and Whitin model for major and minor demands (Hwang et al, 2008). Silver and Meal proposed a heuristic method that finds the optimum order quantity, minimizing the storage and delivery costs (Silver and Meal, 1973). Robinson et al consider coordinated lot-size problems, their variants and exact and heuristic solutions approaches. Interested readers may refer to review article by Robinson et al (Robinson et al, 2009). With the growing focus on supply chain management, firms realize that inventories across the entire supply chain can be more efficiently managed through greater cooperation and better coordination (Arshinder et al., 2008). Integration concept in inventory management models was first proposed by Goyal. In his model the objective was to minimize both the buyer’s and supplier's costs simultaneously in one model (Goyal, 1976). In such models the supplier demand is dependent on the buyer demand. Cohen and lee developed an integrated supply chain model for determining the material requirements strategy (Cohen and Lee, 1988).

Gyana and Bhaba proposed a model for one entity and its objective was to minimize the inventory costs of ingredients and finished products, simultaneously (Gyana and Bhaba, 1999). Ganeshan proposed an ordering point model for minimizing the overall logistics cost of retailers and warehouse (Ganeshan, 1999). Yang and Wee developed an integrated inventory model for buyers and sellers in which the products are assumed to be deteriorative (Yang and Wee, 2000). Hung and others developed an integrated inventory model to determine the optimal inventory policy under conditions of order-processing cost reduction and permissible delay in payments (Huang et al, 2009). Saharidis and others propose a model for comparing Centralized versus decentralized production planning. Two plants are considered, that the product of one plant is input of other plant (Saharidis et al., 2006). Ben et al presented a comprehensive review of the integrated inventory model and also provided some extensions of this important problem. Interested readers may refer to review article by Ben-Daya et al (Ben-Daya et al, 2008).

This paper is different from other similar inventory management researches in two aspects. Firstly, raw material supplier of some company products is in it. Secondly, the effect of using product organizational structure on inventory costs is reviewed in a dairy company.

METHODOLOGY

H_1: "Integrated inventory management within a business structure, has less cost in comparison with independent inventory management in dependent products". To examine this H_1 an integrated inventory management model is developed. The model considers the inventory costs and related constrains. Also convexity is examined and optimizing will be done. The model does sensitivity analysis with different values of input parameters. For independent state, EPQ model will be used with the same parameters and then compared.

Problem definition

The increase in the product diversity and size of many manufacturing companies have caused complications in management of these organizations. In responding problem solving, many companies have changed their organizational structure. Fully authorized business unit form, based on product groups, is a way for concurring this problem. For example, establishing independent business units based on product families (such as, cheese, drinking products and etc) helped KALEH to respond to its customers more rapidly. Business units are Responsible for new product

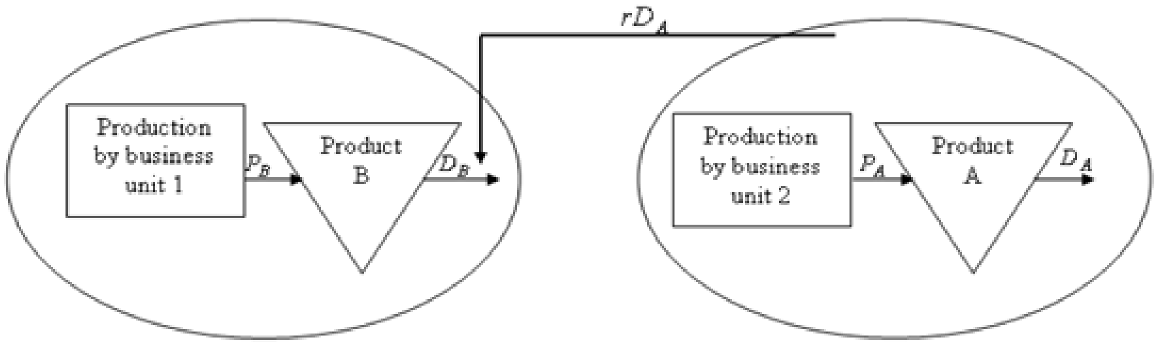

Figure 1. Independent inventory system in business units.

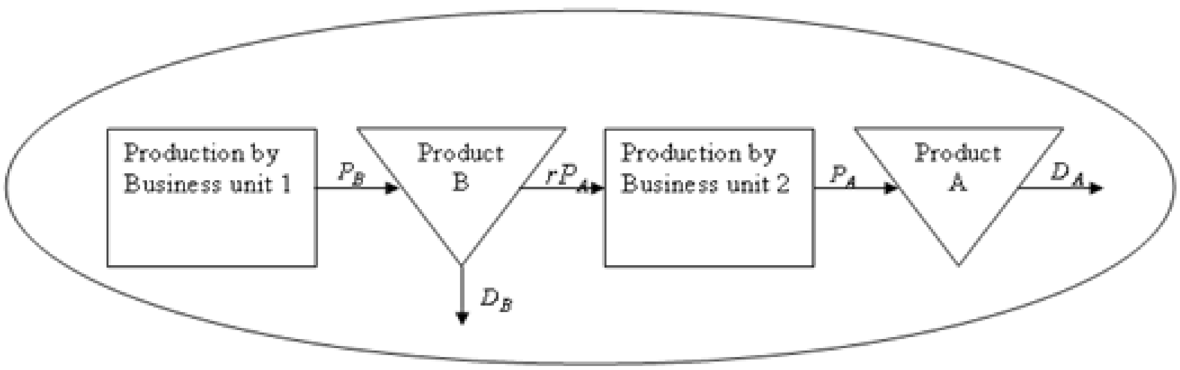

Figure 2. Integrated inventory system in business units.

development, manufacturing, marketing and selling final product.

KALEH's problem now is making decision about products inventory management approach, because most of its products which are in various families have two roles. They have their own market demand and meanwhile they could act as raw material for some other products. For example yogurt is widely supplied in the market as a dairy product but this product is one of the basic raw materials of another dairy product which is in drink family and called "DOUGH". In such condition, inventory management of final product can be done by two approaches. First is inventory management by each business unit independently and second is inventory management integratedly.

The aim of this paper is how to help managers to choose right mechanism for inventory management. In follow are presented variables and parameters that used in this paper.

Model parameters and variables

D_A : Demand rate for product "A" in a unit of time,

D_B : Demand rate for product "B" in a unit of time,

P_A : Production rate for product "A" in a unit of time,

P_B : Production rate for product "B" in a unit of time,

h_A : Inventory holding cost for product "A" in a unit of time,

h_B : Inventory holding cost for product "B" in a unit of time,

K_A : Setup cost of production for product "A",

K_B : Setup cost of production for product "B",

Q_A : Economic production quantity for product "A",

Q_B : Economic production quantity for product "B",

m : Times of producing "A" during one time production of "B",

K : The period in which production of "B" is ceased and its consumption starts,

T : Time horizon for optimization,

t : Time horizon for producing "A",

r : Consumption rate of product "B" in product "A",

As shown in Figure 1, the structure of the system is based on independent model which is the default model of the system. In this model, the final product of business unit one is the raw material of business unit 2l As a result, unit 1 must produce sufficient material for unit 2 to fulfill "A" product demand. So we can conclude that business unit 1 demand is $D_B + rD_A$ in which r is the consumption rate of "B" product in "A" product.

Figure 2 embodies integrated model which is highly recommended. In this case, during production of product "A", product "B" is taken from the stock, based on production rate and also consumption rate of product "B" in manufacturing product "A". Both models of independent and integrated approaches are presented herein after.

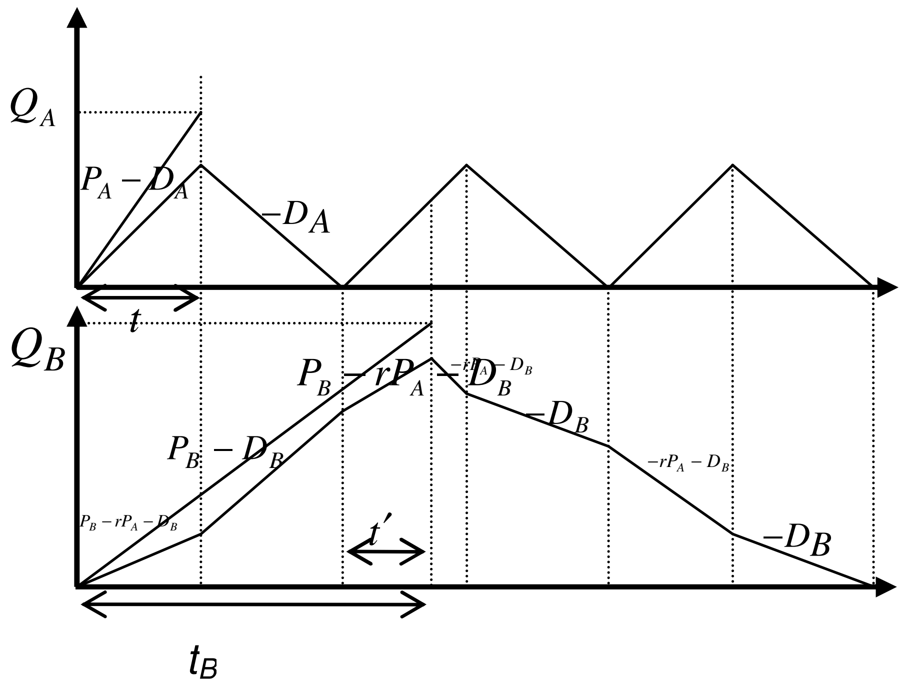

Figure 3. Inventory behavior in integrated model when producing product "B" is finished during producing product "A".

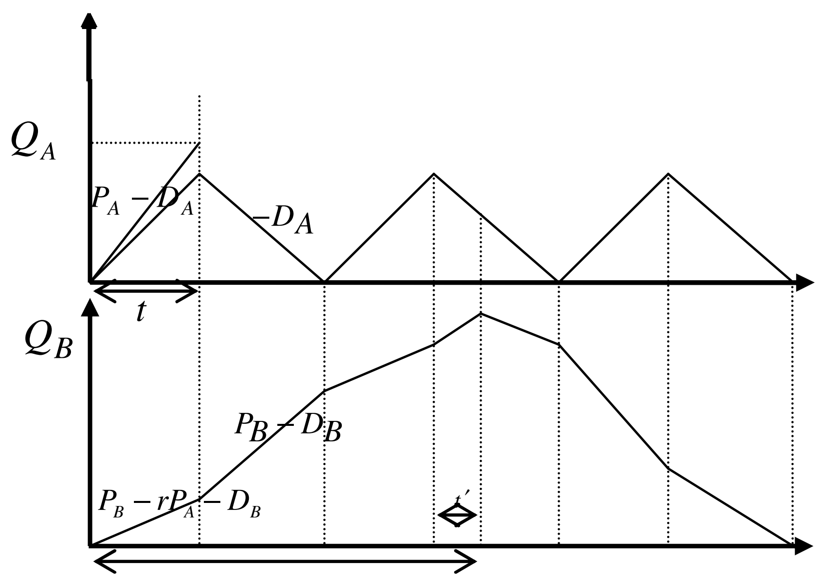

Figure 4. Inventory behavior in integrated model when producing product "B" is finished during consumption product "A"

Integrated inventory model

In this part, the integrated inventory model, presented in figures 3 and 4, is developed. Figures 3 and 4 depict the inventory behavior in the warehouse of "A" and "B" products, when the three-phase production of "A" happens during the single-phase production of "B". In Figure 3, we see that production of "B" is finished during the production of "A". In Fig. 4 the production of "B" is finished during the consumption of "A".

During the production process of "A" and "B", the slope of producing B is $P_B - D_B - rP_A$. This is because "B" has an independent demand in the market. Meanwhile, it is an ingredient of "A". By finishing the production of "A", the aforementioned slope increases to $P_B - D_B$. This process continues till "B" reaches its EPQ. During the consumption phase of "B" and production phase of "A" the slope of consumption of "B" is $-D_B - rP_A$. This is because "B" must supply not only its own demand, but also "A" demand. During the consumption of A this slope reaches $-D_B$.

Assumptions

(a) Demands for products "A" and "B" are deterministic and known.
(b) Cost parameters for "A" and "B" are known constants.
(c) Shortage in "A" and "B" is not permitted.
(d) Production rates of "A" and "B" are greater than their demands rates. Also the production rate of product "B" is bigger than the demand for product "B" plus the required amount of product "B" for producing product "A" ($P_B \geq D_B + rP_A$).
e) Consumption rate of product "B" for producing product "A" is r .

Model discussion

As the production of the product "B" can be finished during either the production or the consumption of product "A", Figures 2 and 3 shows two different states of "B" production process:

a) The process will be finished during producing "A".
b) The process will be finished while consuming "A".
These two states will be described as follows:

1. First state: Based on Figure 3, it is assumed that the production of product "B" is finished in one of the production phases of product "A". Taking into consideration that the setup and holding costs are major components of the objective function, their calculation is described hereinafter.
2. Setup Costs: Equation 1 is utilized for calculating the setup cost for product "A". This equation is based on the fact that there are m cycles of "A" production during one cycle of "B" production.

$$S_1 = mK_A \qquad 1$$

During the planning period (T), product "B" has one time run. This is the premise for calculating the setup cost for product "B", using Equation (2).

$$S_2 = K_B \qquad 2$$

We can reach total setup costs through Equation (3).

$$S = mK_A + K_B \qquad 3$$

3. Holding Costs: We can reach the holding cost of product "A", using the total sum of areas of *m* triangles in fig.3.This is represented in Equation 4.

$$H_A = \frac{mh_A}{2}\left(\frac{t(P_A - D_A)T}{m}\right) \qquad 4$$

The holding cost of product "B" can be calculated considering the area under the inventory behavior in Figure 3.These calculations are described in Appendix 1. Total holding cost of product "A" and "B" are as follow:

$$H = H_A + H_B \qquad 5$$

Total cost and model constraints

Total cost function is a sum function of holding and setup costs. Equation 6 presents this function per each time unit.

$$TC = \frac{(H + S)}{T} \qquad 6$$

Equation 9 shows the supply constraint. In this equation β_n (Appendix 1 equation 26) is the inventory level of product "B" after the production is stopped and τ is the amount of product "B" needed during the period that the production of product "B" is stopped and the demand is responded from the inventory.

$$\varphi = (rP_A + D_B)t + \frac{D_B(P_A - D_A)t}{D_A} \qquad 7$$

$$\tau = (m-k)\varphi + (rP_A + D_B)(t - t_B + (k-1)\frac{T}{m}) \qquad 8$$

$$\beta_n - \tau = 0 \qquad 9$$

Concerning the assumption of first state that the production of product "B" finishes during the production of product "A", constraint 10 is necessary for solving the model.

$$t' - t \leq 0 \qquad 10$$

Finally, the final model for the first state of the problem is as follows:

$$\begin{aligned} &Min \ \ \text{TC} \\ &\text{st:} \\ &\beta_n - \tau = 0 \\ &t' - t \leq 0 \qquad\qquad 11 \\ &t' \geq 0 \\ &k, m \ \ \text{integer} \end{aligned}$$

Second state: In this situation it is assumed that production of product “B” is stopped while the consumption of product “A” is on-going. In this case all costs are as case one and only the holding cost of product “B” is different. The calculations are described in Appendix 1. Therefore, total cost per each time unit is:

$$TC = \frac{S + H_A + H_{B_1} + H_{B_2} + H'_{B_3}}{T} \quad 12$$

The model constraints will be changed in case that the production is stopped during consumption of product “A”. Equation 13 shows the supply constraint in this case.

$$\tau = (m-k)\varphi + D_B((k-1)\frac{T}{m} + t + \frac{(P_A - D_A)t}{D_A} - t_B) \quad 13$$

Having done the aforementioned modifications, the total model changes to the phrase 14.

$$\begin{aligned} & Min \ \text{TC} \\ & \text{st}: \\ & \beta_n - \tau = 0 \\ & t' - \frac{(P_A - D_A)t}{D_A} \leq 0 \\ & t' \geq 0 \\ & k,m \ \text{ integer} \end{aligned} \quad 14$$

Determining T,t' variables, we need to determine m,k, simultaneously. We have utilized numerical method for doing so which is discussed next in this paper.

Total cost optimization algorithm

Since the presented model is convex (see Appendix 2) the following search method is used for reaching the optimum solution.

Step 0 – determine parameters P_A, D_A, P_B, D_B and cost factors K_A, K_B, h_A, h_B.

Step 1 – Select the problem state. As an example we consider Figure 2 state.

Step 2 – For $m=1, k=1$, find the optimum amounts of T,t', using a computational software. If the problem has a feasible solution go to step 3, otherwise go to step 4.

Step 3 – Find $TC_{m,k}$ using equation (19) and put it in TC_{min}.

Step 4 – Add 1 unit to m and let’s $k=1$ then solve the problem using new m, k. If there is a feasible solution, go to step 5, otherwise go to step 6.

Step 5 – If $TC_{m,k} \leq TC_{m-1,k}$, let’s TC_{min} equals $TC_{m,k}$, otherwise go to step 8.

Step 6 – Add 1 unit to *k* and solve the problem. If there is not a feasible solution, go to step 6-1 otherwise check the condition of: $TC_{m,k} \leq TC_{m,k-1}$. If the condition is held, go to step 7, otherwise go to step 8.

Step 7 – Let’s TC_{min} equals $TC_{m,k}$. If $m > k$ then go to step 6, otherwise go to step 4.

Step 8 - T,t' have their optimum amounts, upon which calculate EOQ.

Independent Inventory model

In this case by using the EPQ model (Tersine, 1994) the economic production quantity is calculated for product “A” and product “B”. In this case the demand for product “A” is equal to and the demand for product “B” is equal to. Equation 15 shows the cost function in EPQ model and equation 16 shows the amount of economic production quantity in this model.

$$TC = \frac{DK}{Q} + \frac{HQ(P-D)}{2\text{P}} \quad 15$$

$$Q = \sqrt{\frac{2ADP}{H(P-D)}} \quad 16$$

In the above formula D is the annual demand rate, K is setup cost, P is the annual production rate, H is holding cost and Q stands for order quantity.

Integrated model and inventory model in independent case are compared, using a numerical example, hereinafter.

Numerical example: Comparison of EPQ model and integrated model

In this part a numerical example is presented in order to compare the results of integrated model and EPQ model, Assume that the model parameters are as follows:

$$P_A = 40, \quad D_A = 30, \quad K_A = 3000, \quad h_A = 5,$$
$$P_B = 110, \quad D_B = 60, \quad K_B = 10000, \quad h_B = 5, r = 1$$

For sensitivity analysis of the models, 9 coefficients are determined. These are $1,2,3,4,5,1/5,1/4,1/3,1/2$. Next step is multiplying h_A, K_A, h_B, K_B parameters by the coefficients. Then the total cost of each 2 inventory model is calculated and compared, with each new h_A, K_A, h_B, K_B. Figure 5 shows the total cost changes

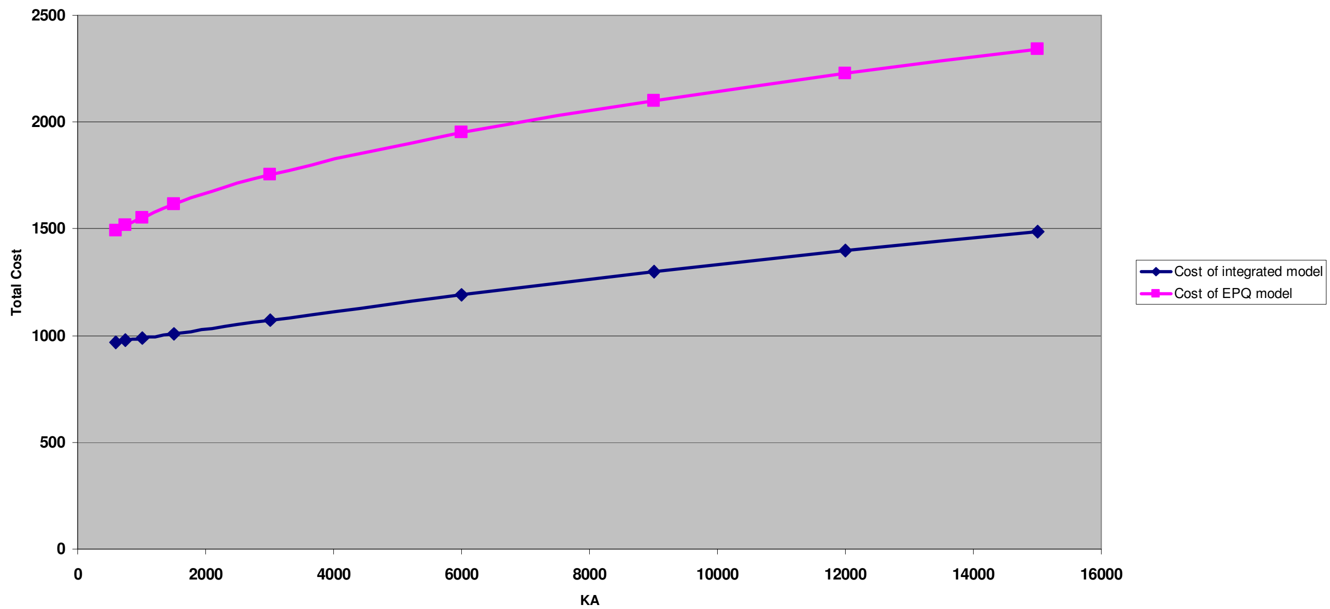

Figure 5. Comparison between integrated and independent in terms of product “A” setup cost.

of EPQ and integrated models compared to setup cost of product “A”. As shown in the graph, by increasing the setup cost of product “A”, the first case of the integrated model has lower cost comparing to EPQ model. The main reason of cost reduction in the integrated model to EPQ model is the same start and end points of inventory cycle for two products of A and B and longer inventory cycle in the integrated model. These two factors cause the reduction of costs in integrated model. In KALEH dairy yogurt and Dough which belong to two different business units are completely separated but starting and finishing the production cycles simultaneously we can reduce total inventory costs.

Figure 6 shows the effect of holding cost of product “A” on the total cost of both integrated and EPQ models. As shown, by increasing the holding cost of product “A”, the integrated model has continuously lower cost compared to EPQ model but the two costs get closer to each other which are caused by shortening of inventory cycle in the integrated model. Since the increase in holding cost of product “A” results shorter inventory cycle of this product , the simultaneous start and end of inventory cycles of product “A” and “B” causes to shorten the inventory cycle of product “B”.

Figures 7, 8 and 9 show the effect of holding cost, setup cost and production rate of product “B” on the total cost of both integrated and EPQ models. As presented, in all cases the total cost in integrated model is lower than EPQ model.

Based on the integrated inventory model results, dependent products have always lower inventory costs comparing to independent state. But if only the holding cost of A product is increasing (supposed that the rest of the costs are fixed) the results of two approaches getting close.

Finally it is suggested that KALEH dairy changes the structure in a way that inventory management of final product will be done integratedly because the total cost is less than independent one. For structure improvement it is proposed that a unit named “ Demand Planning” will be designed to coordinate business units.

Conclusions

Organizational structure based on family products and allocate each family product to a business unit will increase responsiveness in competitive market. This structure is widely used in dairy companies. For instance KALEH is using family product structure. As a result, new product development has improved so that the company can meet customers’ needs easily. The main problem of KAEH’s is how to choose inventory management mechanism.

The root of the problem lays with this reality that there are lots of products in the company in different families which represent not only as final product of one business unit but also act as raw material of other units. For example yogurt family products have their own market but they are an input of drink family production line. In such system, there is an opportunity to have both independent and integrated inventory management approaches. In the first approach, the management is carried out by business units independently and in the second, the inventory management of relative products is carried out in the integrated form.

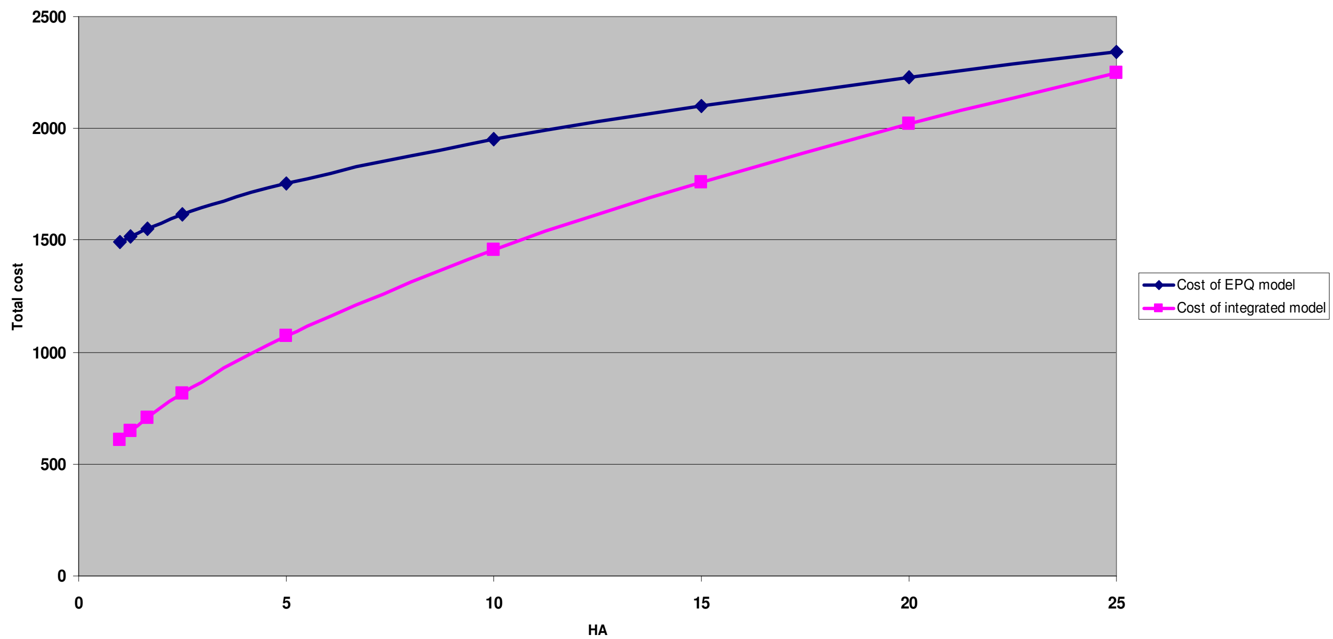

Figure 6. Comparison between integrated and independent model in terms of product “A” holding cost.

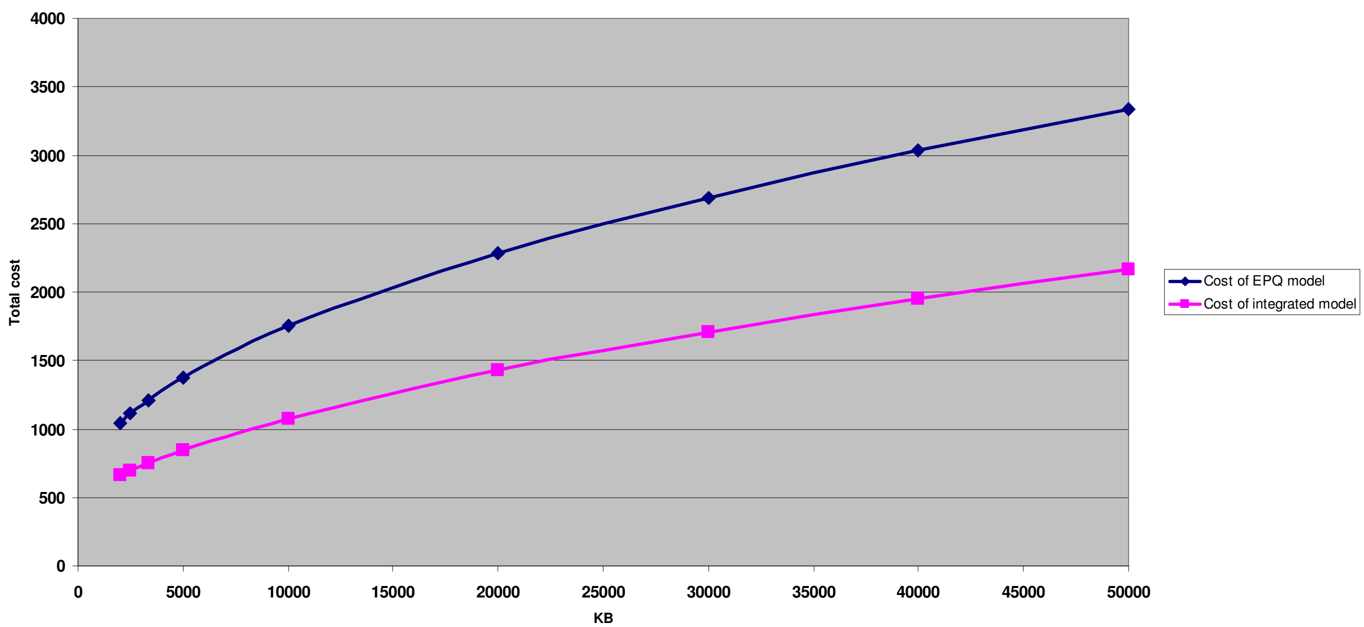

Figure 7. Comparison between integrated and independent model in terms of product “B” setup cost.

In this paper an integrated mathematical model has been developed to choose an appropriate approach and the results have been compared with EPQ model. The findings show that utilizing integrated approach imposes less costs comparing with non integrated approaches.

Considering the approach presented in this paper, there are at least two development aspects for future research. (a) Developing a cost-oriented model to calculate the effect of different organizational structure on inventory cost: Organization structure in production companies has meaningful relation with inventory costs because it determines the centralization or decentralization of company activities. In this paper only the comparison between inventory costs of two structures

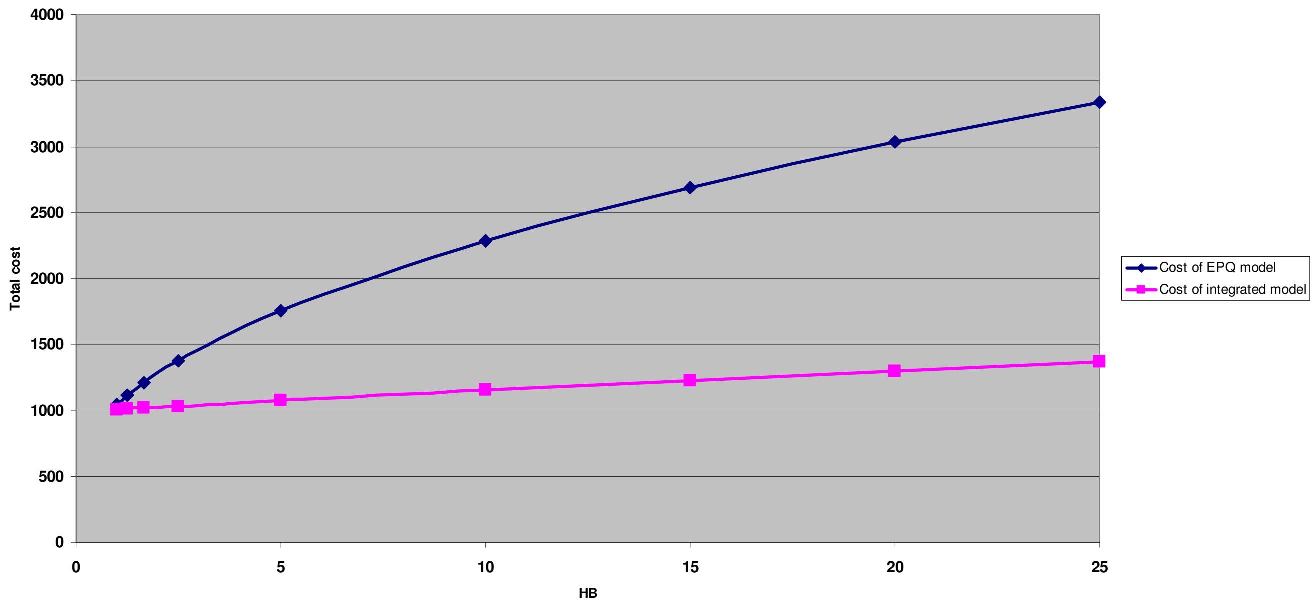

Figure 8. Comparison between integrated and independent models in terms of product “B” holding cost.

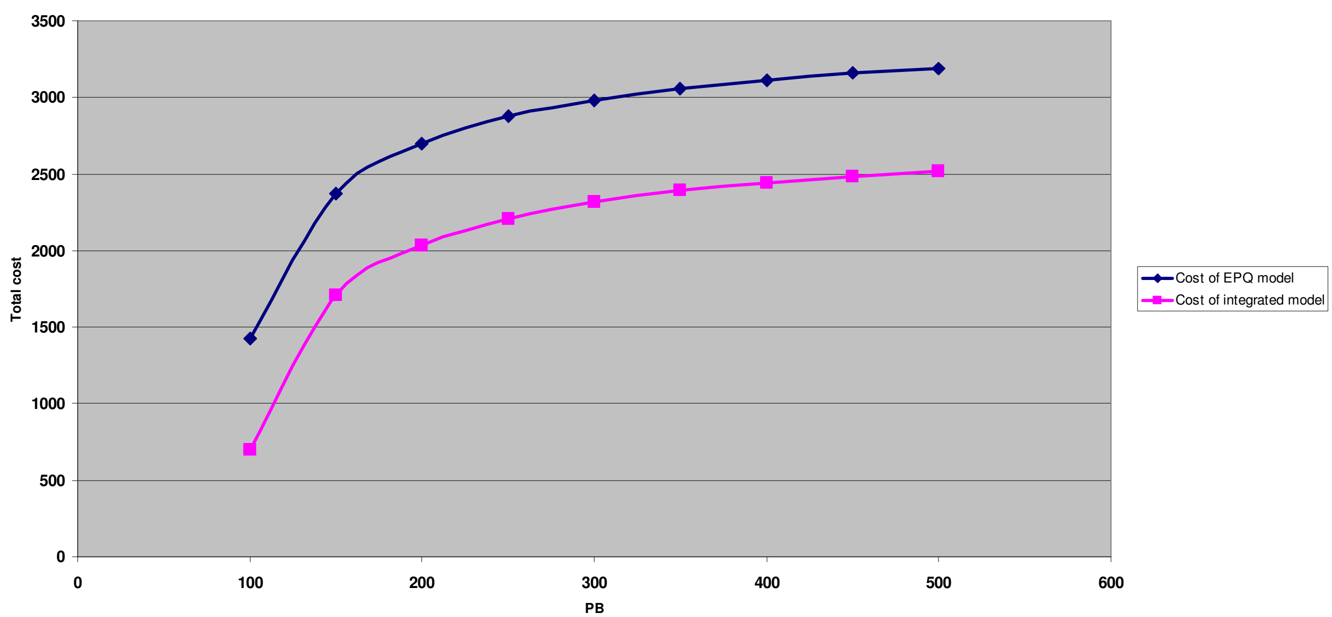

Figure 9. Comparison between integrated and independent models in terms of product “B” production rate.

was done, but it is possible to develop a general model to compare inventory costs of different structures.
(b) Developing an Inventory model considering product return: In such condition, the integrated model in this paper can be developed for both “A” and “B” return.

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Appendix 1

Reaching the holding cost for product "B", we need to calculate the area of fig.3. Doing so, Equations 17, 18, 19 and 20 can be utilized.

$$M_1 = (P_B - D_B - rP_A)t \tag{17}$$

$$N_1 = (\frac{T}{m} - t)(P_B - D_B) \tag{18}$$

$$M_2 = (rP_A + D_B)t \tag{19}$$

$$N_2 = (\frac{T}{m} - t)D_B \tag{20}$$

As indicated in Figure 2, the production of product "B" continues till the *k*th triangle of product "A" (Production phase of product "A"). Afterwards the demands for product "B", which are market demand and demand of product "A", are supplied through the inventory of product "B". During the production phase of product "B", there are *k-1* triangles of product "A"and the inventory volume rate of product "B" is increasing. For each of these triangles, there is an area for product B. These areas have increasing rate till *k-1*th triangle, making an arithmetical progression with common difference of phrase 21.

$$\frac{(M_1 + N_1)T}{m} \tag{21}$$

Therefore for calculating the area of product "B" curve, we utilize the sum of arithmetical progressions, presented in Equation (22).

$$H_{B1} = h_B(\frac{k-1}{2})\left(\frac{kM_1T}{m} + \frac{kN_1T}{m} - \frac{N_1T}{m} - M_1t - N_1t\right) \tag{22}$$

Production-based area for product "A" is calculated through *m-k* triangles. This is in consumption phase of product "B" and is calculated using another arithmetical progression with common difference of phrase 23.

$$\frac{(M_2 + N_2)T}{m} \tag{23}$$

We can reach the production-based area of *m-k* triangles, taking in to account the holding cost, through Equation 24.

$$H_{B2} = h_B(\frac{m-k}{2})\left(N_2(\frac{T}{m} - t) + t(2N_2 + M_2) + (m-k-1)(N_2 + M_2)\frac{T}{m}\right) \tag{24}$$

*k*th triangle representing the time when the production of product "B" is finished during the production of product "A" and the consumption of product "B" starts. The production area of *k*th triangle can be calculated using Equations 25, 26, 27 and 28.

$$\beta_{2k-2} = (k-1)M_1 + (k-1)N_1 \tag{25}$$

$$\beta_n = \beta_{2k-2} + (P_B - rP_A - D_B)(t_B - (k-1)\frac{T}{m}) \tag{26}$$

$$\beta_{2k-1} = \beta_n - (rP_A + D_B)(t - t_B + (k-1)\frac{T}{m}) \tag{27}$$

$$\beta_{2k-1} = \beta_n - (rP_A + D_B)(t - t_B + (k-1)\frac{T}{m}) \tag{28}$$

Taking into account the holding cost, in addition to Equations 25-28, the production-based area for *k*th triangle is reached using Equation 29.

$$H_{B3} = h_B((t_B - (k-1)\frac{T}{m})\beta_{2k-2} + (\beta_n - \beta_{2k-2})\frac{(t_B - (k-1)\frac{T}{m})}{2} + (t - t_B + (k-1)\frac{T}{m})\beta_{2k-1} + (\beta_n - \beta_{2k-1})\frac{(t - t_B + (k-1)\frac{T}{m})}{2} + (\frac{T}{m} - t)\beta_{2k} + D_B\frac{(\frac{T}{m} - t)^2}{2}) \tag{29}$$

The holding cost of product "B" is :

$$H_B = H_{B1} + H_{B2} + H_{B3}$$

In second situation that the production of product "B"is stopped while the consumption of product "A" is on-going , the above equations exists and only the following relations are changed.

$$\beta_n = \beta_{2k-1} + (P_B - D_B)(t_B - (k-1)\frac{T}{m} - t) \tag{30}$$

$$\beta_{2k-1} = kM_1 + (k-1)N_1 \tag{31}$$

$$\beta_{2k} = \beta_n - D_B(k\frac{T}{m} - t_B) \tag{32}$$

Considering the Equations 14 B – 16 B, the production area of *k*th triangle is as follows.

$$H'_{B3} = h_B(t\beta_{2k-2} + M_1\frac{t}{2} + \beta_{2k-1}(t_B - (k-1)\frac{T}{m} - t) + (\beta_n - \beta_{2k-1})\frac{(t_B - (k-1)\frac{T}{m} - t)}{2} + \beta_{2k}(k\frac{T}{m} - t_B) + (\beta_n - \beta_{2k})\frac{(k\frac{T}{m} - t_B)}{2}) \tag{33}$$

Appendix 2: Model Convexity

Concerning the fact that all the constraints in the model are linear, the feasible area is convex. Now, we need to prove the convexity of objective function for all points in feasible area, which depends on phrase 34 (Bazaraa et al., 2006).

$$(t',T)H\begin{pmatrix} t' \\ T \end{pmatrix} \geq 0 \qquad 34$$

Where, *H* represents Hessian matrix.

Considering phrase 34, we reach equation 35.

$$(t',T)\begin{bmatrix} \frac{2mK_A + 2K_B - h_B t'^2 P_B}{T^3} & \frac{h_B t' P_B}{T^2} \\ \frac{h_B t' P_B}{T^2} & -\frac{h_B P_B}{T} \end{bmatrix}\begin{pmatrix} t' \\ T \end{pmatrix} = \frac{2mK_A + 2K_B}{T}$$

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As the amounts of m, K_A, K_B, T are always positive, phrase 34 will be correct and objective function is also convex.