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CHAOS IN PRODUCTION PLANNING

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ABSTRACT. A phenomenon which is seen in some of the manufacturing systems and production planning is chaos and the butterfly effect. The butterfly effect points out that in case of the presence of nonlinear relations in system and incorrect estimation of initial values of variables, the error in the estimates of system state will be intensified, and after a while there will be a large distance between available state of system and reality. Using mathematical means and computer simulation, we have tried to demonstrate that in a production system the numerical combination of Cycle Time (CT), Adjustment Time between existing and desired Work In Progress (WIP), and Adjustment Time between current and desired inventory can lead to chaos and butterfly effect in the behavior of the inventory state variable. Our paper concludes with a discussion of a hypothesis that emerged from this research.

AMS Mathematics Subject Classification :
Key words and phrases : Butterfly effect, chaos, system dynamics, nonlinear dynamics, production systems, bifurcation

1. Introduction

Little research has been carried out in the field of nonlinear dynamics in production systems and the chaos and butterfly effects in such systems. The butterfly effect points out that in case of the presence of nonlinear relations in system and incorrect estimation of initial values of variables, error in the estimate of the system will be intensified, and after a while there will be a large distance between available state of system and reality[3]. The butterfly effect compels us to doubt conventional statistic approaches which compromise with concept of bounded error[4]. The main focus of previous research has been in the following domains:

a. Distinguishing the factors in production systems which result in nonlinear dynamics[1, 2].

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b. Analyzing the causes of chaos in production systems with the use of simulation and mathematical means[8].
c. Expanding some criteria for measuring the complexity and chaos in production systems[14].
d. Expanding planning models in chaotic state[5, 13, 15].

In [1, 2], the following factors are considered as causes of dynamics in production systems:

- Structure
- Order release
- Capacity
- Queuing policies
- Operational rules

Peters et al.[7] have examined capacity in a switched system as one of the causes of chaos. Their main hypothesis is that the capacity of the buffers is limited; therefore the policies should be so that they prevent the buffers from overflowing. The authors have concluded that in return for low values of capacity, chaos is created in the switched system.

Armbruster[7] derived a diagram similar to the bifurcation diagram in the logistic equation after studying bucket brigade systems and considering operational rules as the cause of dynamics in production systems.

While examining the butterfly effects in demand parameters, Wang et al. formulated a strategy for determining production lot sizing proportionate with chaotic demand. The presented method is like that of Wagner's Witin. The only alteration is that the planning periods are considered much smaller. Also, the Lyapunov [13] exponent is used for estimating the degree of chaos.

Efstathiou et al.[9] examine the relation between standards used in measuring complexity in manufacturing systems and supply chains. The authors' main emphasis is on standards of information and chaos theories.

This article is different from the research summarized above in two respects: Firstly, no specific system is considered for demonstrating chaos and the butterfly effect; on the contrary some fixed rules are applied which exist in most production systems. Secondly, system dynamics methodology is used for examining the butterfly effect and chaos.

2. Problem formulation

The primary modeling and analysis tool used in this research is system dynamics (SD) methodology. Forrester [10] introduced SD in the early 60's as a modeling and simulation methodology for long-term decision-making in dynamic industrial management problems. In this part Sterman's[12] adjusted dynamic model is used to model the production system in supply chain. The main logic of Sterman's presenting this model is the same as Jones' in presenting a parametrical method in production planning. Since labor force has no influence on our final aim, which is discovering chaos, in this part we will consider the model independent of labor force. Introducing the variables and parameters:

I : Inventory
WIP : Work In Progress
PSR : Production Start Rate
WAT : WIP Adjustment Time
DPS :Desired Production Start Rate
CT : Cycle Time
IC : Inventory Covering
IAT : Inventory Adjustment Time
DI :Desired Inventory
FT : Forecasting Demand
α : Forecasting Parameter
PR : Production rate
SR : Shipment Rate AFW : Adjustment For WIP
DP : Desired Production
$DWIP$: Desired WIP
$MOPT$: Minimum of Production Time
$RAFI$: Rate Adjustment For Inventory
DIC : Desired Inventory Covering
$Actual$: Actual Demand
$Production\ consists\ of\ two\ subparts$:Inventory and Work In Progress.

It is supposed that there are two main stocks in a production system, and based on production system strategies goods are cast into or taken from these two sources. These two sources are Inventory and Work In Progress. In other words if a product is not in the final stock, i.e. Inventory, it is in the production process, i.e. in the source of Work in Progress (WIP). Production system strategies and external factors such as demand determine the state of trading products between these two sources as well as their fluctuation level. The relations of variables and parameters of the mentioned system are illustrated in the following diagram:

In the Figure5 diagram the production rate (PR) increases the inventory level because this variable is the cause of the product entrance into the stock. On the other hand, considering the demand forecast, the shipment rate (SR) variable is the cause of the product exiting from the source. Thus, the main two factors in fluctuation in inventory are production rate and shipment rate variables. Yet because every inventory needs to save a desired inventory of product considering the situation of the market and factory, we need an adjustment variable in the system to cover the maladjustment in rate adjustment for inventory in case there is a gap between the inventory level and desired inventory level. As shown in the diagram, the production rate variable is the factor that connects the inventory level in work in progress (WIP) and inventory. Generally, product is turned from WIP into inventory when its production process is over, or in other words when it has passed the production cycle time. Like inventory, the source of the product in work in progress should also be at a desired level, because the

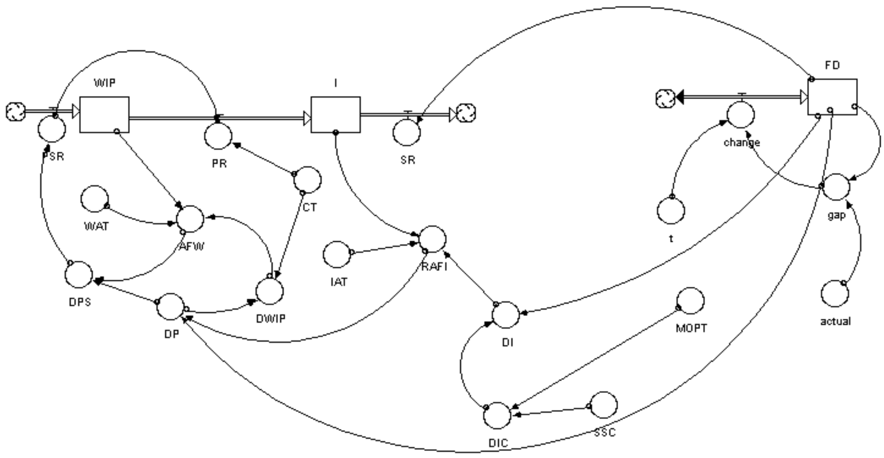

FIGURE 1. The flow diagram of a production system

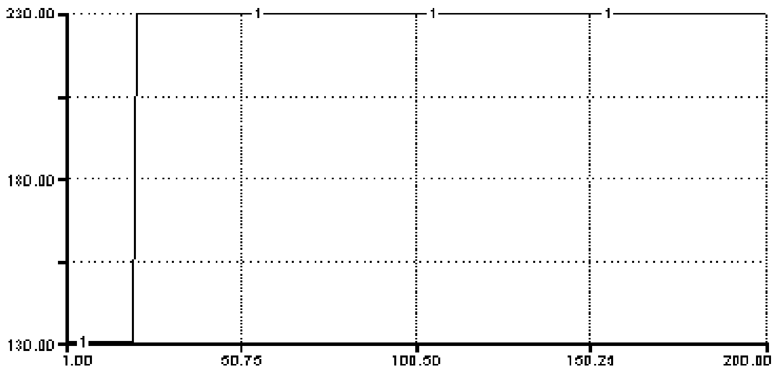

FIGURE 2. Actual demand

shortage of a product in production line results in production stoppage. Thus, an adjustment variable is needed to cover the maladjustment of AFW, taking into account available WIP and desired WIP and the measured time of WAT.

3. Examining "chaos" in a dynamic model

$$DPS = DP + AFW \tag{1}$$
$$PSR = DPS \tag{2}$$

$$\begin{aligned}
SR &= FD && (3)\\
change &= \frac{gap}{t} && (4)\\
gap &= actual - FD && (5)\\
RAFI &= \frac{DI - I}{IAT} && (6)\\
AFW &= \frac{DWIP - WIP}{WAT} && (7)\\
DWIP &= CT \times DP && (8)\\
DP &= Max(0, RAFI + FD) && (9)\\
DIC &= SSC \times MOPT && (10)\\
PR &= \frac{PSR}{CT} && (11)\\
DI &= DIC + FD && (12)\\
FD(t) &= FD(t - dt) + (change) \times dt && (13)\\
I(t) &= I(t - dt) + (PR - SR) \times dt && (14)\\
WIP(t) &= WIP(t - dt) + (PSR - PR) \times dt && (15)
\end{aligned}$$

Equation 9 expresses that the desired product of a production system should be so that it answers the forecasted demand and compensates for the maladjustment of inventory. On the other hand, equation 1 considers a value for desired production rate that includes maladjustment of WIP as well as the desired production. In equation 2 production start rate is considered equal with its desired value (which was calculated in equation 1). Equations 4, 5 and 13 perform the demand forecast using exponential smoothing method[11]; equation 13 is used to get the result of the forecasting. It should also be mentioned that the actual demand used in this paper is shown in illustration 2. In equation 3 it has been supposed that shipment rate, as the inventory reducer factor, equals the forecasted demand. Equations 6 and 7 express that the gap between inventory and desired inventory, and the gap between WIP and its desired value should be covered in IAT and WAT. Equation 8 is the Little Equation, which expresses that the desired WIP value equals the cycle time multiplied by desired product[16]. Equation 10 is used to keep inventory level in a desired level during the delivery time. Equation 11 expresses that in order for a piece to be produced, it should pass the cycle time. Equations 14 and 15 are inventory level equation and WIP level equation.

4. Examining "chaos" in a dynamic model

In this part we examine chaos in the model using mathematical means and computer simulation. After performing several different simulations, it was concluded that under some conditions the numerical combination of the three parameters IAT, CT and WAT causes butterfly effect and chaos in the inventory.

Simulations results divided all the possible instances of chaotic behavior into 8 below situations:

Situation.1: $WAT, CT, IAT \geq 1$

Situation.2: $WAT, IAT \geq 1$ and $CT < 1$

Situation.3: $WAT, CT \geq 1$ and $IAT < 1$

Situation.4: $CT, IAT \geq 1$ and $WAT < 1$

Situation.5: $CT \geq 1$ and $WAT, IAT < 1$

Situation.6: $IAT \geq 1$ and $WAT, CT < 1$

Situation.7: $WAT \geq 1$ and $IAT, CT < 1$

Situation.8: $WAT, IAT, CT < 1$

Since simulation and mathematical confirmation for 8 situations are similar to each other, simulation and mathematical confirmation for situations 1 and 3 are presented completely and for the rest of situations results are presented in conclusion section.

Nonlinear dynamical systems often exhibit chaos, which is characterized by sensitive dependence on initial values or more precisely by a positive Lyapunove exponent. The idea of Lyapunov exponents is to define characteristic numbers for a dynamical system that allow to classify the behaviour of the system in a concise manner. These numbers should account for exponential convergence or divergence of trajectories that start close to each other [6]. Notice that the simplest formula for calculating Lyapunov exponent is:

$$\frac{| I_{n+1} - I_n |}{|I_1 - I_0|}$$

Situation 1. Hypothesis: $Max(0, RAFI + FD) = RAFI + FD$

To analyze the butterfly effect in the model the initial value of variable is considered to be the inventory . The Lyapunov exponent is calculated by changing I to $I + \Delta I$. The initial conditions are judged by analyzing the Lyapunov exponent about system sensitivity.

Proof.

$$\begin{aligned}
I_0 &= I \rightarrow I + \Delta I \\
PR &= \frac{FD + \frac{DI-I}{IAT} + \frac{CT \times [\frac{DI-I}{IAT} + FD] - WIP}{WAT}}{CT} \\
&= \frac{FD}{CT} + \frac{DI - I}{IAT \times WAT} + \frac{DI - I}{IAT \times CT} + \frac{FD}{WAT} - \frac{WIP}{WAT \times CT} \\
SR &= FD \\
I_1 &= I + (PR - SR) \\
I_1 &= I + \frac{FD}{CT} + \frac{DI - I}{IAT \times WAT} + \frac{DI - I}{IAT \times CT} + \frac{FD}{WAT} - \frac{WIP}{WAT \times CT} \\
&\quad -FD
\end{aligned}$$

$$I \rightarrow I + \Delta I \Rightarrow \Big[\frac{FD}{CT} + \frac{DI - I - \Delta I}{IAT \times WAT} + \frac{DI - I - \Delta I}{IAT \times CT} + \frac{FD}{WAT} - \frac{WIP}{WAT \times CT}\Big]$$

□

The effect of changing I to $I + \Delta I$ in WIP is as follow:

$$WIP_{NEW} = WIP_{OLD} + [\frac{-\Delta I}{IAT} - \frac{CT \times \Delta I}{IAT \times WAT} + \frac{\Delta I}{IAT \times CT} + \frac{\Delta I}{IAT \times WAT}]$$

The value inside the brackets shows amount of difference resulted from the change. According to Equation 13 we have:

$$\begin{aligned}
I_1 &= I + (PR - SR) \\
I_2 &= I + \Delta I + (PR' - SR) \\
|I_2 - I_1| &= |\Delta I + \frac{FD}{CT} + \frac{DI - I - \Delta I}{IAT \times CT} + \frac{DI - I - \Delta I}{IAT \times WAT} + \frac{FD}{WAT} - \frac{FD}{CT} \\
&- \frac{DI - I}{IAT \times CT} - \frac{DI - I}{IAT \times WAT} - \frac{FD}{WAT} + \frac{\Delta I}{IAT \times WAT \times CT} \\
&+ \frac{CT \times \Delta I}{IAT \times WAT^2 \times CT} - \frac{\Delta I}{IAT \times WAT \times CT^2} \\
&- \frac{\Delta I}{IAT \times WAT^2 \times CT}| \\
&= |\Delta I - \frac{\Delta I}{IAT \times CT} - \frac{\Delta I}{IAT \times WAT} + \frac{\Delta I}{IAT \times CT \times WAT} \\
&+ \frac{\Delta I}{IAT \times WAT^2} - \frac{\Delta I}{IAT \times WAT \times CT^2} - \frac{\Delta I}{IAT \times WAT^2 \times CT}|
\end{aligned}$$

To calculate the Lyapunov exponent, we divide the above expression into ΔI.

$$\begin{aligned}
&= |1 - \frac{1}{IAT \times CT} - \frac{1}{IAT \times WAT} + \frac{1}{IAT \times CT \times WAT} \\
&+ \frac{1}{IAT \times WAT^2} - \frac{1}{IAT \times WAT \times CT^2} - \frac{1}{IAT \times WAT^2 \times CT}|
\end{aligned}$$

Lyapunov exponent equals Ln in the above expression. After simplifying the product we get:

$$\lambda = Ln|1 - \frac{CT \times WAT^2 + CT^2 \times WAT - CT \times WAT - CT^2 + WAT + CT}{IAT \times CT^2 \times WAT^2}| \tag{16}$$

If we verify that the product of the modulus expression is smaller than one, it will be concluded that in the first situation the system is not chaotic, because the Lyapunov exponent is negative.

Lemma 1. *If* $|1 - X| \leq 1$ *then* $0 \leq X \leq 2$

The fractional expression in modulus is positive because:

$$\begin{aligned} CT &\times WAT^2 > CT \times WAT \\ CT^2 &\times WAT > CT^2 \end{aligned}$$

If we prove that the fractional expression in modulus is equal or smaller than 2, we can conclude that the system is not capable of producing butterfly effect.

Theorem 1. *Suppose:*$A = IAT \times CT^2 \times WAT^2$

$$-1 \leq \frac{WAT - CT \times WAT}{A} \leq 0 \tag{17}$$

$$0 < \frac{CT \times WAT^2}{A} \leq 1 \tag{18}$$

from (17) and (18) we have:

$$0 < \frac{WAT - CT \times WAT + CT \times WAT^2}{A} \leq 1 \tag{19}$$

On the other hand:

$$\begin{aligned} CT^2 &\geq CT \\ 0 \quad < \quad & \frac{CT - CT^2}{A} \leq -1 \end{aligned}$$

$$\rightarrow 0 < \frac{CT^2 \times WAT - CT^2 + CT}{A} \leq 1 \tag{20}$$

Now from (19) and (20) we have:

$$\mid 1 - \frac{CT \times WAT^2 + CT^2 \times WAT - CT \times WAT - CT^2 + WAT + CT}{IAT \times CT^2 \times WAT^2} \mid \leq 1$$

Therefore the Lyapunov exponent cannot be positive, so in this situation the system is not chaotic. Now suppose $Max(0, RAFI + FD) = RAFI + FD$. We will have:

$$\begin{aligned} PR &= \frac{-WIP \times CT}{WAT \times CT} \\ I_1 &= I - (\frac{WIP}{WAT \times CT} - FD) \\ I_2 &= I + \Delta I - (\frac{WIP}{WAT \times CT} \\ &\Rightarrow \frac{|I_2 - I_1|}{\Delta I} = 1, Ln|1| = 0 \end{aligned}$$

Therefore increasing ΔI is not effective.

Situation 3.

Table 1. Two example

Group 1			Group 2		
$\lambda = 2.39$			$\lambda = -0.197$		
CT=1	WAT=0.1	IAT=1	CT=2	WAT=0.9	IAT=4

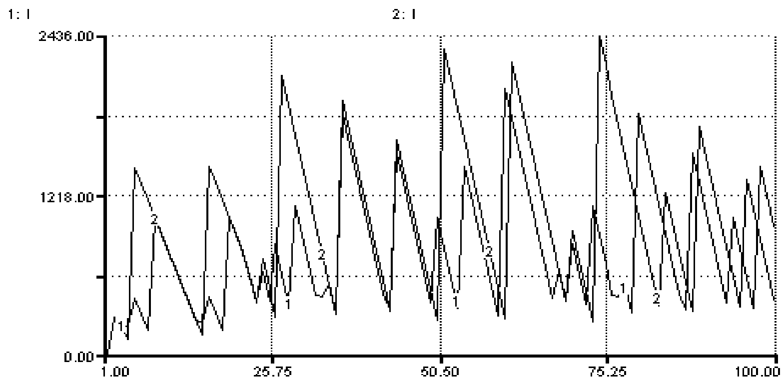

Figure 3. Time diagram of inventory. In the red diagram, the initial value of inventory is zero. In the blue diagram the initial value of inventory is one

a. Mathematical analysis: Hypothesis: $Max(0, RAFI + FD) = RAFI + FD$ Because $CT > 1$, we can use Equation (16) that is given in situation one to analyze the problem.

$$\lambda = Ln|1 - \frac{CT \times WAT^2 + CT^2 \times WAT - CT \times WAT - CT^2 + WAT + CT}{IAT \times CT^2 \times WAT^2}|$$

In this situation also we cannot derive a general rule about the sign of the Lyapunov exponent, because $WAT < 1$ causes a reduction in the denominator. Thus, whether the Lyapunov exponent is negative or positive depends on the numerical combination of IAT, CT and WAT. Two series of example parameter values tabulated in Table 1. For the numbers in group one the Lyapunov exponent will be positive, and for numbers in group two the Lyapunov exponent will be negative.

b.Simulation: The result of the simulation for the values in the first group is written below. As you see, for a change of 1 in the initial value of I, the two diagrams find a great distance which is evidence of the butterfly effect.

The attractor related to this situation is suggested by the phase plot, depicted in Figure 4.

The Figure 5 shows simulation for the values in the second group. As noted in the mathematical analysis, the system is not sensitive to its initial values.

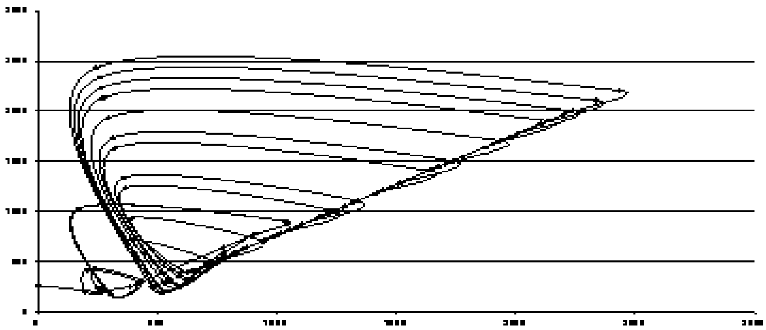

Figure 4. Attractor for the values in group one of situation four

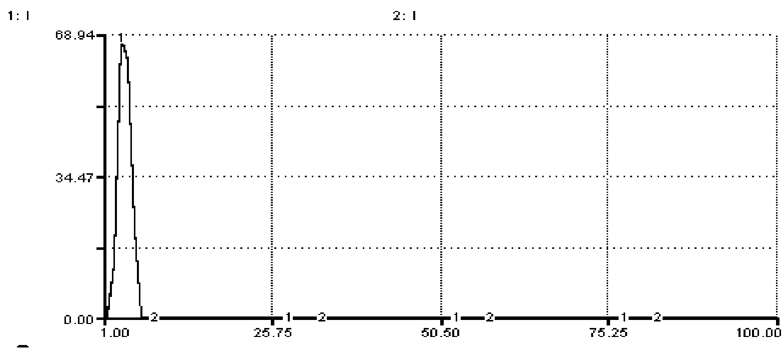

Figure 5. Time diagram for the values in group two of situation four. As you see, a change of one in inventory does not make a great difference.

5. Conclusion

This paper has examined the capability of trigger chaos in a dynamic model of a production system. After performing different simulations, it was concluded that in some situations the numerical value of the parameters of adjustment time between desired and current inventory (IAT), cycle time (CT), and adjustment time between desired and current WIP (WAT) can result in chaos. Thus, the numerical combination of the three parameters was divided into 8 different situations considering whether it was smaller or greater than 1. These divisions are elaborated in the third part of the essay. Using mathematical confirmation and computer simulation, the capability of forming chaos was analyzed in any of the 8 mentioned situations.

If $WAT, CT, IAT \geq 1$, there will be no butterfly effect and chaos in the system, because the Lyapunov exponent is negative, as was shown in the first situation.

In the second situation in which $WAT, IAT \geq 1$ and $CT < 1$, the mathematical confirmation verified that the Lyapunov exponent will always be negative, therefore there would be no butterfly effect. In the other 6 situations, chaos in the system cannot be explained using mathematics. In these situations the existence or nonexistence of chaos in system depends on the numerical combination of the three parameters IAT, WAT, and CT. Generally, it can be said that the requisite condition for existence of butterfly effect is that one of the two parameters IAT or WAT should be smaller than 1. Their being smaller than one shows a kind of acceleration in system to cover maladjustments. In all the situations in which the system shows chaos, the drawn attractors are similar, and the differences lie upon the reduction and increases in phase space and spatial position, and this shows a sort of order in disorder.

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