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An Integrated Inventory Model for Two Products with Internal and External Demands

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Abstract: In this study we have proposed an integrated inventory model for two products having fixed demand. One of the aforementioned products is assumed to be an ingredient for the other product and having an independent demand in the market. The output of the proposed model is the economic level for producing the products with minimum costs. Considering the relations among parameters in the model, the Total cost optimization algorithm for finding the optimum amount of the variables is discussed in addition to a numerical example and its sensitivity analysis. Based on the sensitivity analysis, when one of the products is playing two roles, as indicated above, any change in cost-based and non-cost-based parameters can affect the decision variables of both products. The proposed model is usable for various industries as dairy, pipe and cloths manufacturing.

Key words: Inventory model . Integrated . Economic production level

INTRODUCTION

In many industries, some of the products have two distinct roles. On one side, they are the finished product which can be supplied to the market and on the other side, they can also play the role of an ingredient or a semi-finished product for producing other finished products. This is a common feature for products in dairy and pipe-manufacturing industries. As an example, in dairy industry, yogurt is a finished product which has a definite demand in the market, however it can also be an ingredient for producing Dough, which is a Persian non-alcoholic drink resulted from fusion of yogurt, water, salt and some other additives. In inventory management literature, the inventory systems have been classified based on the dependency and independency of the products demand. As some examples, MRP is an inventory system when we have a dependent demand and Order Point System is the one for independent demand [1]. Each of aforementioned systems has been studied and developed by various researchers, some of which we will mention next in this study.

MRP is an Information System which is used for Managing the inventory and scheduling the ordering of products with dependent demand [2]. One of the main woes in MRP and other inventory systems is determining the order amount which is called lot sizing determination . The response to this question will be the input of MRP and production scheduling systems, when the demand is dependent and independent, respectively. A lot of work has been done in order to find an appropriate response to this question. First model of this kind is called Economical Order Quantity (EOQ) and referred to Harris. This model has been developed based on a fixed demand assumption [3]. A simple expansion to EOQ, the Economic Production Quantity (EPQ) is reached. In this model, the product is assumed to be received or produced gradually and not at once [4]. The aforementioned models have also been developed for the conditions when there is a backordering and shortage [5]. In recent years, the researchers have made the EOQ model more appropriate for the real world by releasing some of its unrealistic assumptions. Among them we can mention: Salameh and Jaber that omitted the assumption of equality of the quality of all received orders [6], Tsou that took in to account the quality costs [7] and Wee *et al*. [8] that considered the inequality of orders quality and also the shortage of orders simultaneously [8]. In many systems, the demand doesn't present a monotonous behavior and varies from period to period. Under such conditions, using static models will make considerable errors. Wagner and Whithin introduced a dynamic programming model in which the demand is a function of time [9]. Silver and Meal proposed a heuristic method that finds the optimum order quantity, minimizing the storage and delivery costs [10].

There is another category of inventory management models in which the model deals with more than one entity and the objective is minimizing the costs or maximizing the profits of all entities,

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simultaneously. Integration concept in inventory management models was first proposed by Goyal. In his model the objective was to minimize the costs of both buyer's and supplier's, simultaneously in one model [11]. In such models the supplier demand is dependent on the buyer demand. Cohen and lee developed an integrated supply chain model for determining the material requiremzents strategy [12]. Gyana and Bhaba proposed a model for one entity and its objective was to minimize the inventory costs of ingredients and finished products, simultaneously [13]. Ganeshan proposed an ordering point model for minimizing the overall logistics cost of retailers and warehouse [14]. Yang and Wee developed an integrated inventory model for buyers and sellers in which the products are assumed to be deteriorative [15].

Gnonia *et al*. [16] present a case study from the automotive industry. This study deals with lot sizing and scheduling problem of a multi-site manufacturing system with capacity constraints and uncertain multi-product and multi demand [16]. Lee and kim propose a hybrid approach combining the analytic and simulation model for production-distribution planning in supply chain, considering capacity constraints [17]. Byrne and baker study a hybrid algorithm combining mathematical programming and simulation models of manufacturing system for the multi-period and multi-product production planning problem [18].

Saharidis *et al*. [19] propose a model for comparing Centralized versus decentralized production planning. Two plants are considered, that the product of one plant is input of other plant [19].

Our paper contributes to the literature in two aspects: firstly, the discussed integration is an intra-firm integration, in which there are two products which not only have independent market demand, but also one of them can be the ingredient of the other one . Secondly , using a proposed mathematical model, we discuss the applicability of transforming the finished products to each other. This model can be utilized in several industries as dairy, cloths and pipe production.

PROBLEM DEFINITION

Assume two products as A and B. Product B not only is an ingredient for product A, but also has independent demand in the market. As an example in dairy industry, yogurt can have the role of B and "Dough", which described earlier in this paper and is derived from yogurt, can have role of A. Under such conditions in which the main input of a product is another product of the firm, the inventory management strategies of the latter product will have a definite effect on the former one. In such a situation, it is necessary to utilize an integrated model for determining the economic production quantities (EPQ) of products A and B. Figure 1 presents the structure of such an integrated model.

Assumptions: Demands for products A and B are deterministic and known.

- Cost parameters for A and B are known constants.
- Shortages in A and B is not permitted.
- Production rates of A and B are greater than their demands rates. Also the production rate of B is greater than sum of production rate of A and demand of B.
- Consumption rate of B for producing A is one to one (one unit of B is utilized in producing one unit of A).
- There are m runs for product A over one run of product B.

Model parameters and variables

D_A: Demand rate for A in a unit of time,
D_B: Demand rate for B in a unit of time,
P_A: Production rate for A in a unit of time,
P_B: Production rate for B in a unit of time,
h_A: Inventory holding cost for product A in a year,
h_B: Inventory holding cost for product B in a year,
K_A: Setup cost of production for A,

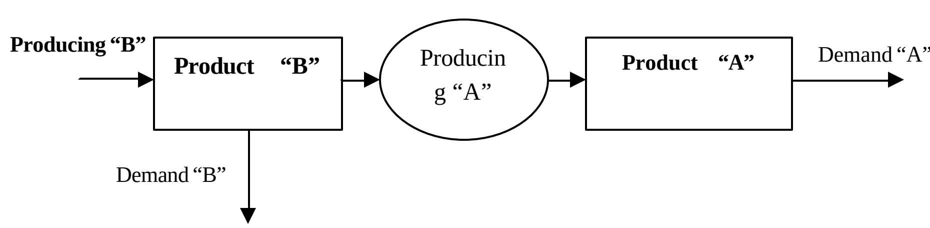

Fig. 1: The inventory model structure for product A and product B products

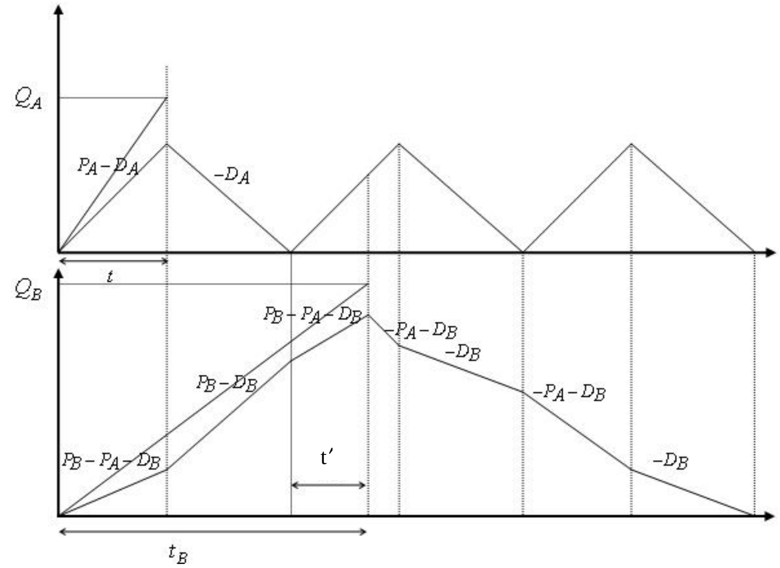

Fig. 2: Producing product "B" is finished during producing product "A"

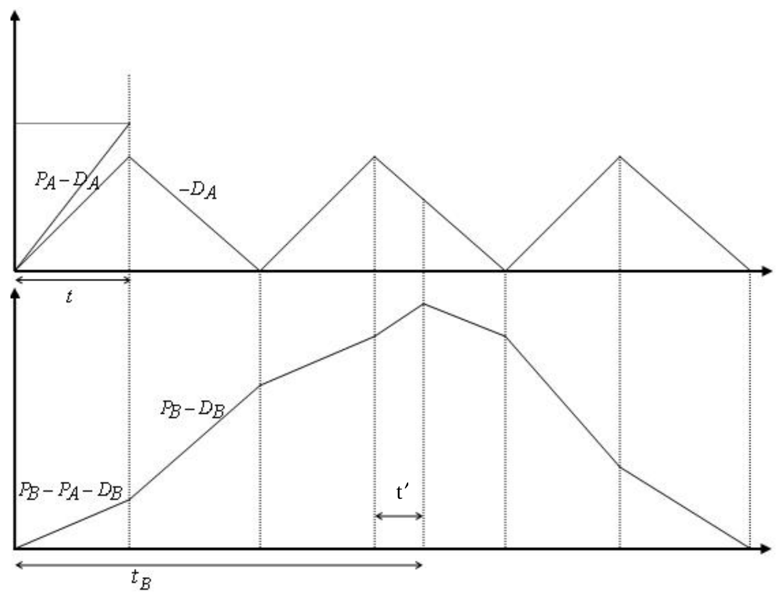

Fig. 3: Producing product "B" is finished during consumption phase of product A

K_B: Setup cost of production for B,
Q_A: Economic production quantity (EPQ) for A,
Q_B: Economic production quantity (EPQ) for B,
m: Times of producing A during one time production of B,
K: The period in which production of B is ceased and its consumption starts,
T: Time horizon for optimization,
t: Time horizon for producing A,

Figure 2 depicts the inventory behavior in the warehouse of A and B products, when three times production of A happens during one time production of B. In this Figure, we assume that production of B is finished during the production of A. In Fig. 3 we assume that the production of B is finished during the consumption of A.

When both products are produced the inventory level of product B increases with rate P_B-D_B-P_A. This is because B has an independent demand in the market and in addition is an ingredient for producing A. When the production of A ceases, the aforementioned slope increases to P_B-D_B. This process continues till B reaches its EPQ. During the consumption phase of B and during the production phase of A the slope of consumption of B is $-D_B-P_A$. Because it is necessary for B to supply not only its own demand, but also A demand. During the consumption of A this slope reaches $-D_B$.

MODEL DISCUSSION

As the production of the product B can be finished during either the production or the consumption of product A, we have considered two various states for developing proposed model in. we will discuss these states, which have been presented in Fig. 2 and 3.
Setup and holding costs are the main elements of the objective function.

Setup costs: Calculating setup cost for the case of Fig. 2 and 3 are the same. We utilize equation 1 for calculating the setup cost for product A. This equation is based on the fact that based on our assumptions there are *m* times of runs for product A.

$$S_1 = mK_A \tag{1}$$

During the planning period (T), product B has one time run. This is the premise for calculating the setup cost for product B, using Equation (2).

$$S_2 = K_B \tag{2}$$

We can reach total setup costs through Equation (3).

$$S = mK_A + K_B \tag{3}$$

Holding costs: We can reach the holding cost of product A, using the total sum of areas of *m* triangles in Fig. 2 and 3. This is represented in Equation 4.

$$H_A = \frac{mh_A}{2}\left\{(t(P_A - D_A)\frac{T}{m}\right\} \tag{4}$$

Reaching the holding cost for product B, we need to calculate the area of Fig. 2 and 3. See appendix.
Therefore total holding cost can be calculated using equation 5.

$$H = H_A + H_B \tag{5}$$

Total cost and model constraints: Total cost function is a sum function of holding and setup costs. Equation 6 presents this function per each time unit.

$$TC = \frac{(H + S)}{T} \tag{6}$$

Most important constraint of the model is the fact that the stocked inventory of product B, after finishing its production, must be enough to supply product A and product B market demands. This constraint can be presented using equations 14 (see appendix) and 7-9:

$$\phi = (P_A + D_B)t + \frac{D_B(P_A - D_A)t}{D_A} \tag{7}$$

$$\tau = (m - k)\phi + (P_A + D_B)(t - t_B + (k - 1)\frac{T}{m}) \tag{8}$$

$$\beta_n - \tau = 0 \tag{9}$$

Concerning the assumption of first state that the production of product B finishes during the production of product A, constraint 10 is necessary for solving the model.

$$t' - t \le 0 \tag{10}$$

Finally, the final model for the first state of the problem (Fig. 2) is as follows:

$$\begin{aligned} &\text{Min } TC \\ &\text{st:} \\ &\beta_n - \tau = 0 \\ &t' - t \le 0 \\ &t' \ge 0 \\ &k, m \ \text{ integer} \end{aligned} \tag{11}$$

In the state of Fig. 3, product B production finishes during the consumption phase of product A, the equation 8 must also be modified, as equation 12.

$$\tau' = (m - k)\phi + D_B((k - 1)\frac{T}{m} + t + \frac{(P_A - D_A)t}{D_A} - t_B) \tag{12}$$

Having done the aforementioned modifications, the total model changes as follow.

$$\begin{aligned} &\text{Min } TC \\ &\text{st:} \\ &\beta'_n - \tau' = 0 \\ &t' - \frac{(P_A - D_A)t}{D_A} \le 0 \\ &t' \ge 0 \\ &k, m \ \text{ integer} \end{aligned} \tag{13}$$

Determining T, t' variables, we need to determine m, k, simultaneously. We have utilized numerical method for doing so which is discussed next in this paper.

Model convexity: Concerning the fact that all the constraints in the model are linear, the feasible area is convex. Now, we need to prove the convexity of objective function for all points in feasible area, which depends on expression 14 [20].

$$(t',T)H\begin{pmatrix} t' \\ T \end{pmatrix} \geq 0 \qquad (14)$$

where *H* represents Hessian matrix.
Considering expression 14, we reach equation 15.

$$(t',T)\begin{bmatrix} \frac{2mK_A + 2K_B - h_B t'^2 P_B}{T^3} & \frac{h_B t' P_B}{T^2} \\ \frac{h_B t' P_B}{T^2} & -\frac{h_B P_B}{T} \end{bmatrix}\begin{pmatrix} t' \\ T \end{pmatrix} = \frac{2mK_A + 2K_B}{T} \qquad (15)$$

As the amounts of m, K_A, K_B, T are always positive, expression 14 will be correct and objective function is also convex.

TOTAL COST OPTIMIZATION ALGORITHM

As we mentioned earlier, for given parameters and m,k we can easily find the optimal values of T, t'. But the optimal values of integer variables cannot be found through an analytical procedure. So we propose the following simple search procedure to find optimal decision variables.

Step 0: Determine the values of the system parameters P_A, D_A, P_B and D_B, as well as the cost factors K_A, K_B, h_A and h_B.

Step 1: Select the problem state. As an example we consider Fig. 2 state.

Step 2: For m = 1, k = 1, find the optimum amounts of T, t', using a computational software. If the problem has a feasible solution go to step 3, otherwise go to step 4.

Step 3: Find $TC_{m,k}$ using equation (6) and put it in TC_{min}.

Step 4: Add 1 unit to m and let's k = 1 then solve the problem using new m, k. If there is a feasible solution, go to step 5, otherwise go to step 6.

Step 5: If $TC_{m,k} \leq TC_{m-1,k}$, let's TC_{min} equals $TC_{m,k}$, otherwise go to step 8.

Step 6: Add 1 unit to *k* and solve the problem. If there is not a feasible solution, go to step 6-1 otherwise check the condition of: $TC_{m,k} \leq TC_{m,,k-1}$. If the condition is held, go to step 7, otherwise go to step 8.

Step 6-1: If m>k then go to step 6 otherwise go to step 4.

Step 7: Let's TC_{min} equals $TC_{m,k}$. If m>k then go to step 6, otherwise go to step 4.

Step 8: T, t' have their optimum amounts, upon which calculate EOQ.

SENSITIVITY ANALYSIS

Assume the model parameters as below:

$$P_A = 40,\quad D_A = 30,\ K_A = 3000,\ h_A = 5,$$
$$P_B = 110,\ D_B = 60,\ K_B = 10000,\ h_B = 5$$

Based on these parameters, the optimum solution is reached in m = 1, k = 1 and is as follows:

$$Q_A = 331.2,\ Q_B = 1821.6$$

In this part, the system responses is discussed based on the changes in parameters. For each cost parameter, we have considered 9 different levels as: 1/5, ¼, 1/3, ½, 1, 2, 3, 4, 5, which are equal to costs in numerical example.

Figure 4 presents that by changing the setup cost of product A, Q_A, Q_B and *TC* are increased that the slope of EPQ for product B is more than the slope of EPQ for product A. since Increasing in EPQ for product A means increasing the production time of product A and also product B is ingredient of product A. So, we can conclude that the model behavior is close to what happens in reality.

Figure 5 depicts the effect of product A holding cost on EPQs and TCs. As is indicated in Fig. 5, increasing the holding cost of product A, causes the EOQs to decrease. This is mainly because, increase in holding costs leads to increase in warehousing costs in comparison to fixed and setup costs, making the production in smaller batches more economical. As product B is an ingredient for product A, so any decrease in EPQ of product A, will decrease the EPQ of product B as well. Increase in costs is mainly because any decrease in EPQs, will decrease the production

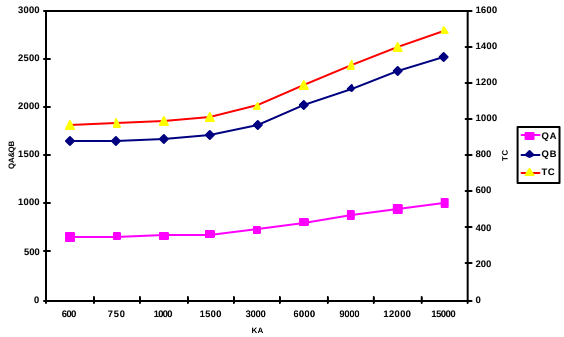

Fig. 4: Effect of K_A on the system

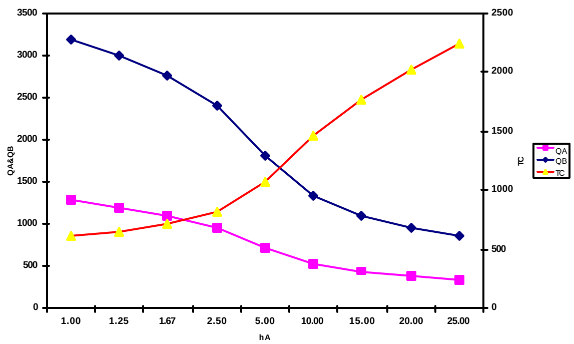

Fig. 5: Effect of h_A on the system

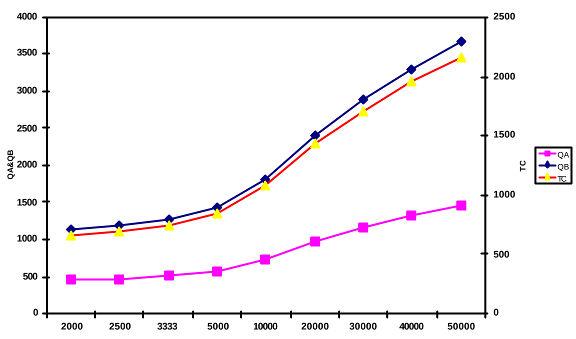

Fig. 6: Effect of K_B on the system

cycle, which increases the setup cost per each unit of time.

Figure 6 presents the effect of changes in setup cost of product B on EPQs and TCs. By increasing the setup cost of product B, the slope of increase in product B EPQ, is greater than the slope of increase in product A EPQ. Here, due to increase in setup cost, it is more economical to produce the product B in larger batches. As product B is also an ingredient for product A, any changes in setup cost of this product will have some effects on product A EPQ.

Figure 7 shows the effect of holding cost of product B on EPQs and TCs. Any increase in holding cost of product B, leads to decreases in product B EPQs and increase in TCs.

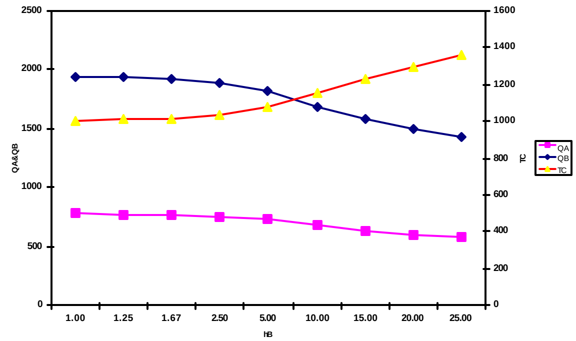

Fig. 7: Effect of h_B on the system

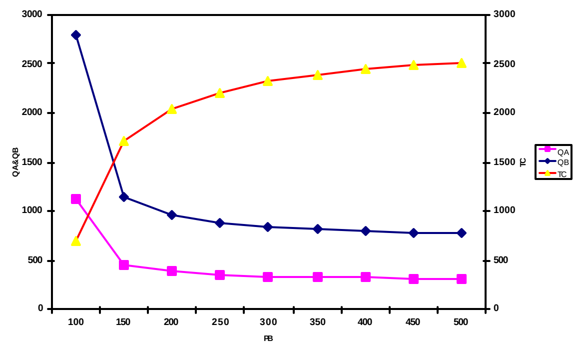

Fig. 8: Effect of P_B on the system

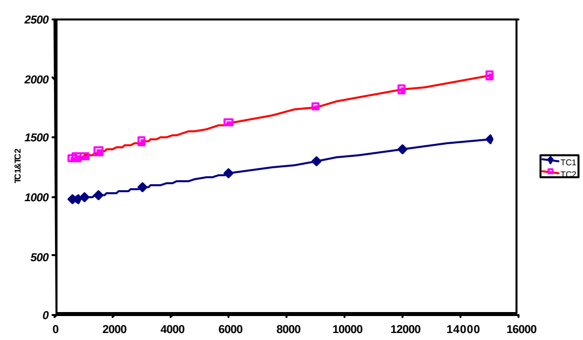

Fig. 9: Comparison between the costs of two states in terms of K_A

Figure 8 depicts the model behavior against the non-cost parameter of production capacity of product B. Increasing the production capacity of product B, causes Q_A and Q_B to decrease and TC to increase with a decreasing slope. Increase in production capacity without any increase in demands, causes the holding cost and so the overall costs to increase to soar.

Figure 2 and 3 present two various states of the problem. As discussed earlier, in the first state, the production of product B, ceases during the production phase of product A, however in the second phase, this happens during the consumption phase of product A. Figure 9-11 present some comparisons between the costs of two states, by changing some cost parameters. As indicated in the figures, as the inventory level in second state is always more than is level in first state, the cost of first state is invariably lower than the cost of second state.

In all aforementioned states, the optimum quantities for k and m are: m = 1, k = 1. Concerning the nine levels we considered for each parameters and the results of sensitivity analysis, the optimum solution will most probably reached in m = 1, k = 1. This problem has been solved for more than 45 various states and always the optimum response is gained in m = 1, k = 1.

CONCLUSIONS AND FUTURE RESEARCH

The proposed model in this study is for analyzing the inventory system with two products, which not only have fixed demands in market, but also one of them is an ingredient for the other one. For simplicity purposes, we have assumed that the productions of both products have been started simultaneously and during one run of the ingredient, several runs of the next product happen. We have developed the model for two various states. The optimum amounts for the decision variables are reached using the solution procedure and software for non-linear programming. Finally, the special features of the model are analyzed, using sensitivity analysis. Results show that when two products are dependent, production wise, any change in cost-based and non-cost-based parameters affects the amount of decision variables in both products.

The result of this paper is applicable for many industries, in which their semi-finished products also have an independent demand in the market. Some examples are dairy, pipe-production and clothes industries.

The discussed concept in this paper can be expanded as:

- Utilizing probability theory for discussing the demands in the market.

Appendix

Calculating the area of Fig. 2, Equation 16, 17, 18 and 19 can be utilized.

$$M_1 = (P_B - D_B - P_A)t \tag{16}$$

$$N_1 = (\frac{T}{m} - t)(P_B - D_B) \tag{17}$$

$$M_2 = (P_A + D_B)t \tag{18}$$

$$N_2 = (\frac{T}{m} - t)D_B \tag{19}$$

As is indicated in Fig. 2, the production of product B continues till the *k*th triangle of product A (Production phase of product A). Afterwards the demands for product B, which are market and product A demands to product B, is supplied through the inventory of product B. During the production phase of product B, there are *k-1* triangles of product A and the inventory volume rate of product B is increasing. For each of these triangles, there is an area for product B. These areas have increasing rate till *k-1*th triangle, making an arithmetical progression with common difference of $\frac{(M_1 + N_1)T}{m}$. Therefore for calculating the area of product B curve, we utilize the sum of arithmetical progressions, presented in equation (20).

$$H_{B1} = h_B(\frac{k-1}{2})\left(\frac{kM_1T}{m} + \frac{kN_1T}{m} - \frac{N_1T}{m} - M_1t - N_1t\right) \tag{20}$$

Production-based area for product A is calculated through *m-k* triangles. This is in consumption phase of product B and is calculated using another arithmetical progression with common difference of $\frac{(M_2 + N_2)T}{m}$. We can reach the production-based area of *m-k* triangles, taking in to account the holding cost, through Eequation 21.

$$H_{B2} = h_B(\frac{m-k}{2})\begin{pmatrix} N_2(\frac{T}{m} - t) + t(2N_2 + M_2) \\ +(m-k-1)(N_2 + M_2)\frac{T}{m} \end{pmatrix} \tag{21}$$

*k*th triangle representing the time when the production of product B is finished during the production of product A and the consumption of product B starts. The production area of *k*th triangle can be calculated using equations 22, 23, 24 and 25.

$$\beta_{2k-2} = (k-1)M_1 + (k-1)N_1 \tag{22}$$

$$\beta_n = \beta_{2k-2} + (P_B - P_A - D_B)(t_B - (k-1)\frac{T}{m}) \tag{23}$$

$$\beta_{2k-1} = \beta_n - (P_A + D_B)(t - t_B + (k-1)\frac{T}{m}) \tag{24}$$

$$\beta_{2k} = \beta_{2k-1} - D_B(\frac{T}{m} - t) \tag{25}$$

Taking into account the holding cost, in addition to equations 22-25, the production-based area for *k*th triangle is reached using equation 26.

$$\begin{aligned} H_{B3} = h_B\Big(& (t_B - (k-1)\frac{T}{m})\beta_{2k-2} \\ & + (\beta_n - \beta_{2k-2})\frac{(t_B - (k-1)\frac{T}{m})}{2} \\ & + (t - t_B + (k-1)\frac{T}{m})\beta_{2k-1} \\ & + (\beta_n - \beta_{2k-1})\frac{(t - t_B + (k-1)\frac{T}{m})}{2} \\ & + (\frac{T}{m} - t)\beta_{2k} + D_B\frac{(\frac{T}{m} - t)^2}{2} \Big) \end{aligned} \tag{26}$$

Finally total holding cost for product B is:

$$H_B = H_{B1} + H_{B2} + H_{B3}$$

In the second state, production of product B finishes during the product A consumption (Fig. 3). In this states the equations 23-25 change to the followings, but the other equations remain the same.

$$\beta'_n = \beta_{2k-1} + (P_B - D_B)(t_B - (k-1)\frac{T}{m} - t) \tag{23}$$

$$\beta_{2k-1} = kM_1 + (k-1)N_1 \tag{24}$$

$$\beta_{2k} = \beta'_n - D_B(k\frac{T}{m} - t_B) \tag{25}$$

Considering the equations 23-25, the production area of *k*th triangle is as follows.

$$\begin{aligned} H'_{B3} = h_B\Big(& t\beta_{2k-2} + M_1\frac{t}{2} + \beta_{2k-1}(t_B - (k-1)\frac{T}{m} - t) \\ & + (\beta'_n - \beta_{2k-1})\frac{(t_B - (k-1)\frac{T}{m} - t)}{2} \\ & + \beta_{2k}(k\frac{T}{m} - t_B) + (\beta'_n - \beta_{2k})\frac{(k\frac{T}{m} - t_B)}{2} \Big) \end{aligned} \tag{26}$$

Finally total holding cost for product B in Fig. 3 is:

$$H_B = H_{B1} + H_{B2} + H'_{B3}$$

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